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A Review Study on Various Forms of Generalized Nano Open Sets in Nano Topology

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Abstract
The purpose of this article is to review most recent research on several kinds of generalized nano open sets in nano topological spaces, such as: nano α-open set, nano semi-open set, nano pre-open set, nano b-open set, nano β-open set and nano somewhere dense set, and then we investigate their implications and properties equipped with counter examples, in addition, we prove some of their related properties in special cases of approximations, and discuss where these class of sets become equivalent.
Keywords: Approximation space, nano topological space and generalizations, interior and clousre in topologcial space.
MSC (2010): Primary 54A05, Secondary 54D10, 54D15.

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عرض لدراسة حول صيغ متعددة من المجموعات النانو المفتوحة المعممة في التبولوجية النانوية

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الملخص

الغرض من هذا البحث هو عرض أحدث المقالات حول أنواع متعددة من المجموعات النانو المفتوحة المعممة في الفضاءات التبولوجية النانوية، مثل: مجموعة النانو المفتوحة من النوع الفا، مجموعة النانو شبه المفتوحة، مجموعة النانو المسبقة، مجموعة النانو المفتوحة من النوع بي، مجموعة النانو المفتوحة من النوع بيتا ومجموعة النانو الكثيفة في مكان ما، و من ثم قمنا بدراسة علاقتها ببعض و خواصها مزودين بأمثلة مضادة، بالإضافة إلى ذلك، قمنا بإثبات بعض الخواص المرتبطة بها في حالات خاصة من التقريبات، و مناقشة أين تصبح تلك المجموعات متكافئة.

الكلمات المفتاحية: الفضاء التقريبي، الفضاء التبولوجي النانوي وعمليات الداخلية والعلاقة في الفضاء التبولوجي.

1. Introduction

The theory of rough sets was due to Pawlak in 1982 [1], which is mainly concerned with the approximation of objects using an equivalence relation on the universe of this approximation space. In 2013 [2] Thivagar and Richard used the ideal of the approximation space to introduced a nano topological space with respect to a subset X of an universe U, which is defined in terms of lower and upper approximations and boundary region, the authors established

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various forms of nearby nano open sets; as nano regular open set, nano α-open set, nano semi-open set and nano pre-open set, and they derived some properties under different cases of approximations. Revathy and Iiango, in 2015 [3] introduced the class of nano β-open sets, and studied their characterizations, and then Nasef and Aggour [4] and Sathishmohani et al [5] investigated more properies on the class of nearby nano open sets, and they defined nano β-interior, nano β-closure, and nano β-boundary and they studied their topological properties and their relations with the classical nano operators. Nano b-open sets were also studied by Parimala et al, in 2016 [6], when they provided some of their characterizations, and recently in 2023, Arwini and Sakah [7] defined the concept of nano somewhere dense set in nano topology, and they proved that this notion is equivalent with nano β-open set, therefor they used the propetries of somewhere dense sets to provide some more properties on nano β-open sets.
In the present paper, we review various forms of generalized nano open sets, such as nano α-open set, nano semi-open set, nano pre-open set, nano b-open set, nano β-open set and nano somewhere dense set, where we illustrate their relations, and then we study the behavior of these generalizations in some special cases regarding the lower and upper approximations.
The paper is divided into four sections as follows; in section two we recall the basic concepts in approximation space and in nano topological space, then in section three we investigate the class of different generalized nano open sets, and illustrate their relations and provide the characterizations of these sets in some special cases in lower and upper approximations, and finally in section four we present our conclusion.

Nano Open Sets in Nano Topology

Definition 2.1 [1] Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Elements belonging to the same equvalence

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class are said to be indiscernible with one another, the pair (U, R) is said to be the approximation space. Let $X \subseteq U$, then:

a) The lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and denoted by $L_R(X)$.

i.e. $L_R(X) = \cup_{x \in U} \{[x]: [x] \subseteq X\}$ where$[x]$ denotes the equvalence class determined by x.

b) The upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and denoted by $U_R(X)$.

i.e. $U_R(X) = \cup_{x \in U} \{[x]: [x] \cap X \neq \emptyset\}$.

c) The boundary region of X with respect to R is the set of all objects, which can be clssified neither as X nor as not -X with respect to R and it is denoted by $B_R(X)$.

i.e. $B_R(X) = U_R(X) - L_R(X)$.

Example 2.2 Let $U = \{x_1, x_2, \dots, x_8\}$ be a universe set, and R be an equivalence relation on U with the following equivalence classes: $E_1 = \{x_1, x_4, x_8\}$, $E_2 = \{x_2, x_5, x_7\}$, $E_3 = \{x_3\}$ and $E_4 = \{x_6\}$,
i.e., $U/R = \{E_1, E_2, E_3, E_4\}$.
If $X = \{x_1, x_4, x_7\}$, then: $L_R(X) = \emptyset$
and $U_R(X) = \{x_1, x_2, x_4, x_5, x_7, x_8\} = B_R(X)$.
If $Y = \{x_3, x_8\}$, then: $L_R(X) = \{x_3\}$, $U_R(X) = \{x_1, x_3, x_4, x_8\}$
and $B_R(X) = \{x_1, x_4, x_8\}$.
If $Z = \{x_3\}$, then: $L_R(X) = U_R(X) = Z$ and $B_R(X) = \emptyset$.

Theorem 2.3 [1] If (U, R) is an approximation space and X, Y are subsets of U, then:

i) $L_R(X) \subseteq X$.
ii) $L_R(\emptyset) = \emptyset$ and $L_R(U) = U$.
iii) $L_R(X \cup Y) \supseteq L_R(X) \cup L_X(Y)$.
iv) $L_R(X \cap Y) \subseteq L_R(X) \cap L_R(Y)$.
v) $L_R(X) \subseteq L_R(Y)$ and $U_R(X) \subseteq U_R(Y)$ whenever $X \subseteq Y$.

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Theorem 2.4 [1] If (U, R) is an approximation space and X, Y are subsets of U, then:

- ***i)*** $X \subseteq U_R(X)$.
- ***ii)*** $U_R(\emptyset) = \emptyset$ and $U_R(U) = U$.
- ***iii)*** $U_R(X) \subseteq U_R(Y)$ whenever $X \subseteq Y$.
- ***iv)*** $U_R(X \cup Y) = U_R(X) \cup U_R(Y)$.
- ***v)*** $U_R(X \cap Y) \subseteq U_R(X) \cap U_R(Y)$.
- ***vi)*** $U_R(X^C) = [L_R(X)]^C$ and $L_R(X^c) = [U_R(X)]^c$.
- ***vii)*** $U_R U_R(X) = L_R U_R(X) = U_R(X)$.
- ***viii)*** $L_R L_R(X) = U_R L_R(X) = L_R(X)$.

Definition 2.5 [2] Let U be the universe, R be an equivalence relation on U and $X \subseteq U$ with the collection:

$$\tau_R(X) = \{U, \emptyset, L_R(X), U_R(X), B_R(X)\}.$$

Then $\tau_R(X)$ is a topology on U called the nano topology on U with respect to X, and the pair $(U, \tau_R(X))$ is called the nano topological space.

Definition 2.6 [2] Let $(U, \tau_R(X))$ be a nano topological space, then any element of the collection $\tau_R(X)$ is called a nano-open set (briefly N-open), while the complement of the nano open set is called a nano-closed set (briefly N-closed), and the family of all nano-closed sets in $\tau_R(X)$ is denoted by $\mathcal{F}_R(X)$. Note that:

$$\mathcal{F}_R(X) = \{U, \emptyset, [L_R(X)]^c, [U_R(X)]^c, [B_R(X)]^c\}.$$

Example 2.7 Let $U = \{x_1, x_2, x_3, x_4, x_5\}$ be a universe set with equivalence relations on U as follows:

If $U/R_1 = \{\{x_i\}: i \in \{1,2,3,4,5\}\}$, then for any $X \subseteq U$ we have:
$\tau_{R_1}(X) = \{U, \emptyset, X\}$ and $\mathcal{F}_{R_1}(X) = \{U, \emptyset, X^c\}$.
If $U/R_2 = \{U\}$, then for any $X \subseteq U$ we have:
$\tau_{R_2}(X) = \{U, \emptyset\} = \mathcal{F}_{R_2}(X)$.
If $U/R_3 = \{\{x_1, x_3, x_5\}, \{x_2\}, \{x_4\}\}$ and $X = \{x_1, x_2\}$, then we have:
$\tau_{R_3}(X) = \{U, \emptyset, \{x_2\}, \{x_1, x_2, x_3, x_5\}, \{x_1, x_3, x_5\}\}$ and

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$\mathcal{F}_{R_3}(X) = \{U, \emptyset, \{x_1, x_3, x_4, x_5\}, \{x_4\}, \{x_2, x_4\}\}$.

Definition2.8 [2] If $(U, \tau_R(X))$ is a nano topological space with respect to X where $X \subseteq U$ and if $A \subseteq U$, then the nano interior of A is defined as the union of all nano-open sets of U that is contained in A, and it is denoted by $Nint(A)$, i.e. $Nint(A)$ is the largest nano-open subset of A.

Theorem 2.9 [2] In a nano topological space $(U, \tau_R(X))$, if F, G are subsets of U, then:

i) F is a nano open set if and only if $Nint(F) = F$.
ii) $Nint(\emptyset) = \emptyset$ and $Nint(U) = U$.
iii) If $F \subseteq G$ then $Nint(F) \subseteq Nint(G)$.
iv) $Nint(F) \cup Nint(G) \subseteq Nint(F \cup G)$.
v) $Nint\ (F \cap G) = Nint(F) \cap Nint(G)$.
vi) $Nint(Nint(F)) = Nint(F)$.

Definition2.10 [2] If $(U, \tau_R(X))$ is a nano topological space with respect to X where $X \subseteq U$ and if $A \subseteq U$, then the nano-closure of A is defined as the intersection of all nano-closed sets containing A, and it is denoted by $Ncl(A)$, i.e. $Ncl(A)$ is the smallest nano-closed set containing A.

Theorem2.11 [2] In a nano topological space $(U, \tau_R(X))$, if F, G are subsets of U, then:

i) $F \subseteq Ncl(F)$.
ii) F is a nano closed if and only if $Ncl(F) = F$.
iii) $Ncl(\emptyset) = \emptyset$ and $Ncl(U) = U$.
iv) If $F \subseteq G$ then $Ncl(F) \subseteq Ncl(G)$.
v) $Ncl(F \cup G) = Ncl(F) \cup Ncl(G)$.
vi) $Ncl(F \cap G) = Ncl(F) \cap Ncl(G)$.
vii) $Ncl(Ncl(F)) = Ncl(F)$.

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Example 2.12 Let $U = \{x_1, x_2, x_3, x_4, x_5\}$ be a universe set with equivalence relation R on U as follows:
$U/R = \{\{x_1\}, \{x_2, x_4\}, \{x_3, x_5\}\}$ and $X = \{x_1, x_3, x_4\}$, then we have:
$\tau_R(X) = \{U, \emptyset, \{x_1\}, \{x_2, x_3, x_4, x_5\}\}$ and $\mathcal{F}_R(X) = \tau_R(X)$.
If $A = \{x_1, x_2\}$, $B = \{x_3, x_4, x_5\}$ and $C = \{x_2, x_3, x_4, x_5\}$, then we have:
$Nint(A) = \{x_1\}$, $Ncl(A) = U$,
$Nint(B) = \emptyset$, $Ncl(B) = \{x_2, x_3, x_4, x_5\}$,
and $Nint(C) = Ncl(C) = C$.

Theorem 2.13 [2] In a nano topoogical space $(U, \tau_R(X))$, we have:

- ***i)*** $Ncl\big(U_R(X)\big) = U$.
- ***ii)*** $Ncl\big(L_R(X)\big) = [B_R(X)]^c$.
- ***iii)*** $Ncl\big(B_R(X)\big) = [L_R(X)]^c$.

Proof. Direct form Definition (2.6), since

$$\mathcal{F}_R(X) = \{U, \emptyset, [L_R(X)]^c, [U_R(X)]^c, [B_R(X)]^c\}.$$

3. Different Generalizations of Nano Open Sets

In this section, we review various forms of generalized nano open sets, as nano α-open set, nano semi-open set, nano pre-open set, nano b-open set, nano β-open set and nano somewhere dense set, where we illustrate their relations, and study the behavior of these generalizations in some special cases regarding the lower and upper approximations.

Definition 3.1 [2,3,6] In a nano topological space $\big(U, \tau_R(X)\big)$, then a subset B in U is called:

- **a)** nano α-open (briefly $N\alpha$-open) if $B \subseteq Nint\big(Ncl\big(Nint(B)\big)\big)$.
- **b)** nano semi-open (briefly NS-open) if $B \subseteq Ncl(Nint(B))$.
- **c)** nano pre-open (briefly NP-open) if $B \subseteq Nint(Ncl(B))$.
- **d)** nano b-open (briefly Nb -open) if $B \subseteq Nint\big(Ncl\ (B)\big) \cup Ncl(Nint(B))$.

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e) nano β-open (briefly $N\beta$-open) if $B \subseteq Ncl(Nint(Ncl(B)))$.

f) nano somewhere dense (briefly NS-dense) if $Nint(Ncl(B)) \neq \emptyset$, where B is non-empty set.

$N\alpha O(U,X)$, $NSO(U,X)$, $NPO(U,X)$, $NbO(U,X)$, $N\beta O(U,X)$ and $NSD(U,X)$ denoted the families of all nano α-open, nano semi-open, nano preopen, nano b-open, nano β-open and nano S-dense subsets of U; respectively.

Theorem 3.2 [3] In a nano topological space $(U, \tau_R(X))$, a non-empty subset A *of* U *is* $N\beta$-open in U if and only if $A \cap U_R(X) \neq \emptyset$.

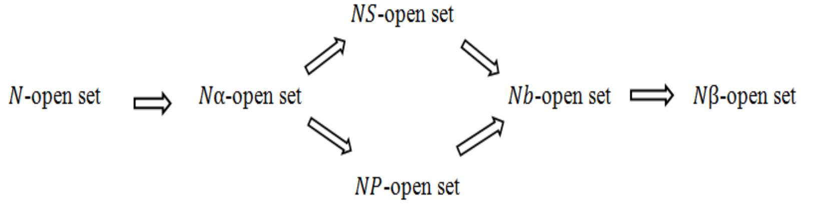
Corollary 3.3 [3] If $U_R(X) = U$ in a nano topological space $(U, \tau_R(X))$, then $N\beta O(U,X) = P(U)$.

Theorem 3.4 [7] In a nano topological space $(U, \tau_R(X))$, if A is a non-empty subset of U, then these statements are equivalent:

i) A is NS-dense.

ii) A is $N\beta$-open set.

Theorem 3.5 [4] The implications between nano open sets and the class of nano generalization open sets are given in this diagram; as follows:



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Examples 3.6 Inverse directions in the prevouis diagram are not true in general, for example suppose $U=\{a,b,c,d,e\}$, $U/R=\{\{a\},\{b,c\},\{d\},\{e\}\}$ and $X=\{a,b\}$, then:
$\tau_R(X)=\{\emptyset,U,\{a\},\{a,b,c\},\{b,c\}\}$, hence
$\mathcal{F}_R(X)=\{\emptyset,U,\{d,e\},\{a,d,e\},\{b,c,d,e\}\}$.

Therefore:

1. The set $\{b,d,e\}$ is Nβ-open set but not Nb-open, since $Nint(\{b,d,e\})=\emptyset,\ Ncl(Nint(\{b,d,e\}))=\emptyset, Ncl(\{b,d,e\})=\{b,c,d,e\},\ Nint(Ncl(\{b,d,e\}))=\{b,c\}$ and $Ncl(Nint(Ncl(\{b,d,e\})))=\{b,c,d,e\}$.
2. The set $\{b\}$ is Nb-open but not NS-open, since $Ncl(\{b\})=\{b,c,d,e\}$, so $Nint(Ncl(\{b\}))=\{b,c\}$, while $Nint(\{b\})=\emptyset$, so $Ncl(Nint(\{b\})=\emptyset$.
3. The set $\{a,d\}$ is Nb-open but not NP-open, since $Ncl(\{a,d\})=\{a,d,e\}$, so $Nint(Ncl(\{a,d\}))=\{a\}$, while $Nint(\{a,d\})=\{a\}$, so $Ncl(Nint(\{a,d\})=\{a,d,e\}$.
4. The set $\{a,b,c,d\}$ is $N\alpha$-open but not N-open, since $Nint(\{a,b,c,d\})=\{a,b,c\}$, so $Ncl(Nint(\{a,b,c,d\})=U$, hence $Nint(Ncl(Nint(\{a,b,c,d\}))=U$.

Definition 3.7 [4] In a nano space $(U,\tau_R(X))$, a subset C of U is called $N\alpha$-closed (NS-closed, NP-closed, Nb-closed and $N\beta$-closed) if its complement is $N\alpha$-open (NS-open, NP-open, Nb-open and $N\beta$-open; resp).

Theorem 3.8 [4] The implications between nano closed sets and the class of nano generalization closed sets are given in this diagram; as follows:

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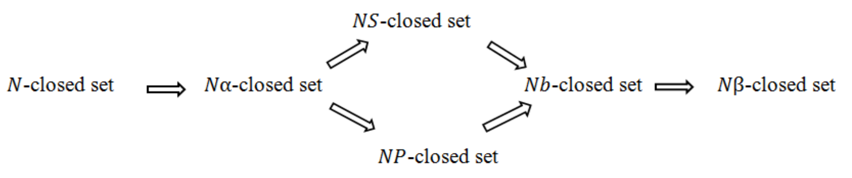
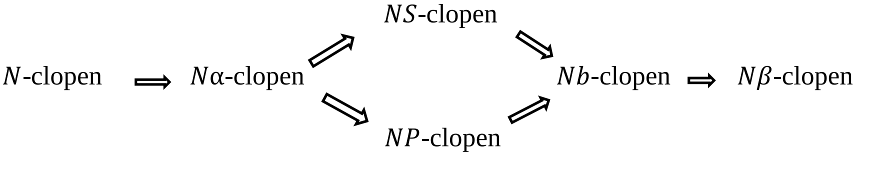


Example 3.9 In the previous example note that: $\{a, e\}$ is $N\beta$-closed, $\{a, c, d, e\}$ is Nb-closed, $\{a, d, e\}$ is NS-closed, $\{c, d, e\}$ is NP-closed and $\{e\}$ is $N\alpha$-closed set.

Definition 3.10 [4] In a nano space $(U, \tau_R(X))$, a subset C of U is called $N\alpha$-clopen (NS -clopen, NP -clopen, Nb -clopen and $N\beta$-clopen) if D and D^c are $N\alpha$-open (NS-open, NP-open, Nb-open and $N\beta$-open; resp).

Theorem 3.11 [4] N-clopen, $N\alpha$-clopen, NS-clopen, NP-clopen, Nb-clopen and $N\beta$-clopen subsets of a nano space $(U, \tau_R(X))$ are ordered as follows:



Examples 3.12 In Example (3.6) note that: $\{a, e\}$ is $N\beta$-clopen, set $\{a, d\}$ is Nb-clopen, $\{a, e\}$ is NS-clopen, and in this example there are no NP-clopen nor $N\alpha$-clopen subsets in U, while in Example (2.12) the sets $\{x_2, x_4\}$ and $\{x_1\}$ are NP-clopen set and $N\alpha$-clopen in U; respectively.

Theorem 3.13 In a nano topological space $(U, \tau_R(X))$, let A be a non-empty subset of U, then:

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i) if D is N-open set and A is a subset of U such that $A \subseteq D$, then A is $N\beta$-open.
ii) if D is N-open set and A is a subset of U such that $A \cap D \neq \emptyset$, then A is $N\beta$-open.
iii) if D be $N\beta$-open set and A is a subset of U such that $D \subseteq A$, then A is $N\beta$-open.

Proof.
i) If $A \subseteq D$ and D is N-open set, then $A \subseteq D \subseteq U_R(X)$, hence is A intersect $U_R(X)$ and by Theorem (3.2) we get A is $N\beta$-open set.
ii) If D is N-open set and A is a subset of U such that $A \cap D \neq \emptyset$, then $D \subseteq U_R(X)$ hence A intersect $U_R(X)$, so A is $N\beta$-open.
iii) If D is $N\beta$-open set, then D intersect $U_R(X)$, and since A is a subset of U that contains D, we get A is $N\beta$-open.

Corollary 3.14 In a nano space $(U, \tau_R(X))$ if A and B are two $N\beta$-open sets in U, then $A \cup B$ is also $N\beta$-open set.
Proof. Since A is $N\beta$-open set and $A \subseteq A \cup B$, and by using (iii) in the previous Theorem we get $A \cup B$ is also $N\beta$-open.

Example 3.15 The intersection of two $N\beta$-open sets is not necessarily $N\beta$-open set, for example the sets $\{a, d\}$ and $\{b, d\}$ are $N\beta$-open sets in the nano space given in Example (3.6), but $\{d\} = \{a, d\} \cap \{b, d\}$ is not $N\beta$-open.

Corollary 3.16 In a nano topological space $(U, \tau_R(X))$, we have:
i) $N\beta O(U, X) = P(U) - \{A \subseteq U: A \subseteq {U_R}^c(X)\}$.
ii) $N\beta C(U, X) = P(U) - \{A \subseteq U: U_R(X) \subseteq A\}$.
iii) The family of all $N\beta$-clopen subsets in U is given by:

$$P(U) - \{A \subseteq U: A \subseteq {U_R}^c(X) \ or \ U_R(X) \subseteq A\}$$
$$= \{\varphi\} \cup \{A \subseteq U: U_R(X) \cap A \neq \varphi \text{ and } {U_R}^c(X) \cap A \neq \varphi\}.$$

Proof. Direct from Theorem (3.2).

Remark 3.17 In a nano topological space $(U, \tau_R(X))$, any subset of U is $N\beta$-open set or $N\beta$-closed set.
Proof. Direct from the previous corollary.

Theorem 3.18 [2] In a nano topological space $(U, \tau_R(X))$:

- ***i)*** if$L_R(X) = \varphi$, then $N\alpha O(U, X) = \{\varphi\} \cup \{A \subseteq U: U_R(X) \subseteq A\}$.
- ***ii)*** if $U_R(X) = L_R(X) = X$, then $N\alpha O(U, X) = \{\varphi\} \cup \{A \subseteq U: L_R(X) \subseteq A\}$.
- ***iii)*** if $U_R(X) = U$ and $L_R(X) \neq \varphi$, then $N\alpha O(U, X) = \tau_R(X) = \{U, \varphi, L_R(X), L_R(X)^c\}$.

Corollary 3.19 If $U_R(X) = U$ in a nano topological space $(U, \tau_R(X))$, then $N\alpha O(U, X) = \tau_R(X)$.
Proof. Direct form the prevouis Theorem (i) and (iii).

Theorem 3.20 [2] In a nano topological space $(U, \tau_R(X))$, $N\alpha O(U, X) = NSO(U, X) \cap NPO(U, X)$.

Theorem 3.21 [2] In a nano topological space $(U, \tau_R(X))$:

- ***i)*** if $L_R(X) = \varphi$ and $U_R(X) \neq U$, then $NSO(U, X) = \{\varphi\} \cup \{A \subseteq U: U_R(X) \subseteq A\}$.
- ***ii)*** If $U_R(X) = L_R(X) = X$, then $NSO(U, X) = \{\varphi\} \cup \{A \subseteq U: L_R(X) \subseteq A\}$.
- ***iii)*** If $U_R(X) = U$, then $NSO(U, X) = \tau_R(X) = \{U, \varphi, L_R(X), L_R(X)^c\}$.
- ***iv)*** If $U_R(X) \neq L_R(X)$, $U_R(X) \neq U$ and $L_R(X) \neq \varphi$, then $NSO(U, X) = \{U, \varphi, L_R(X), B_R(X)\} \cup \{A \subseteq U: U_R(X) \subseteq A\} \cup \{A \subseteq U: A = B \cup L_R(X)$ or $A = B \cup B_R(X)$ where $B \subseteq U_R(X)^c\}$.

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Corollary 3.22 If $U_R(X) = U$ in a nano topological space $(U, \tau_R(X))$, then $NSO(U,X) = N\alpha O(U,X) = \tau_R(X)$.
Proof. Direct from Corollary (3.19) and Theorem (3.21) (i) and (iii).

Theorem 3.23 [2] In a nano topological space $(U, \tau_R(X))$ if A and B are two NS-open sets of U, then $A \cup B$ is also NS-open set.

Example 3.24 The intersection of two NS-open sets is not necessarily NS-open set, for example the sets $\{a, d\}$ and $\{b, c, d\}$ are NS-open sets in the nano space given in Example (3.6), but $\{d\} = \{a, d\} \cap \{b, c, d\}$ is not NS-open.

Theorem 3.25 In a nano topological space $(U, \tau_R(X))$:

i) If $U_R(X) = U$, then $NPO(U,X) = NbO(U,X) = P(U)$.

ii) if $L_R(X) = \varphi$ and $U_R(X) \neq U$, then $NPO(U,X) = NbO(U,X) = \{\varphi\} \cup \{A \subseteq U : A \cap U_R(X) \neq \emptyset\}$.

iii) If $U_R(X) = L_R(X) = X$, then $NPO(U,X) = NbO(U,X) = \{\varphi\} \cup \{A \subseteq U : A \cap U_R(X) \neq \emptyset\}$.

Proof.

i) If $U_R(X) = U$, we have $\tau_R(X) = \{U, \varphi, L_R(X), L_R(X)^c\} = \mathcal{F}_R(X)$, then:

$Ncl(A) = L_R(X)$ wheneve A is a non-empty set such that $A \subseteq L_R(X)$, hence $Nint\big(Ncl(A)\big) = L_R(X)$, therefor A is NP-open set, so it is Nb-open set.
$Ncl(A) = U$ wheneve A is a non-empty set such that $A \cap L_R(X) \neq \varphi$ and $A \cap L_R(X)^c \neq \varphi$, hence $Nint\big(Ncl(A)\big) = U$, therefor A is NP-open set, so it is Nb-open set.
$Ncl(A) = L_R(X)^c$ wheneve A is a non-empty set such that $A \subseteq L_R(X)^c$, hence $Nint(Ncl(A)) = L_R(X)^c$, therefor A is NP-open set, so it is Nb-open set.
Hence we obtain $NPO(U,X) = NbO(U,X) = P(U)$.

ii) If $L_R(X) = \varphi$ and $U_R(X) \neq U$, then $\tau_R(X) = \{U, \emptyset, U_R(X)\}$ and $\mathcal{F}_R(X) = \{U, \emptyset, U_R(X)^c\}$, then:

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$Ncl(A)=U$ wheneve A is a non-empty set such that $A \subseteq U_R(X)$, hence $Nin\big(Ncl(A)\big)=U$, therefor A is NP-open set, so it is Nb-open set.
$Ncl(A)=U$ wheneve A is a non-empty set such that $A \cap U_R(X) \neq \varphi$ and $A \cap U_R(X)^c \neq \varphi$, hence $Nint\big(Ncl(A)\big)=U$, therefor A is NP-open set, so it is Nb-open set.
$Ncl(A)=U$ wheneve A is a non-empty set such that $A \subseteq U_R(X)^c$, hence $Ncl(A)=U_R(X)^c$, so $Nint(Ncl(A))=\varphi$, therefor A is not NP-open set. Moreover $Nint(A)=\varphi$, hence $Ncl(Nint(A))=\varphi$, therefor A isnot Nb-open set.
Hence we obtain $NPO(U,X)=NbO(U,X)=\{\varphi\} \cup \{A \subseteq U: A \cap U_R(X) \neq \varphi\}$.

iii) If $U_R(X)=L_R(X)=X$, then:
in the case when $U_R(X)=L_R(X)=X=U$ then as in case (i) we have any subset of U is NP-open set, so it is Nb-open set.
In the case when $X \neq U$, then $\tau_R(X)=\{U,\emptyset,X\}$ and $\mathcal{F}_R(X)=\{U,\emptyset,X^c\}$, hence:
$Ncl(A)=U$ wheneve A is a non-empty set such that $A \subseteq U_R(X)$, hence $Nint\big(Ncl(A)\big)=U$, therefor A is NP-open set, so it is Nb-open set.
$Ncl(A)=U_R(X)^c=X^c$ wheneve A is a non-empty set such that $A \cap U_R(X) \neq \varphi$and $A \cap U_R(X)^c \neq \varphi$, hence $Nint\big(Ncl(A)\big)=\varphi$, therefor A is not NP-open set. Moreover, $Ncl(Nint(A))=Ncl(\varphi)=\varphi$, so it is not Nb-open set.
$Ncl(A)=U$ wheneve A is a non-empty set such that $A \subseteq U_R(X)^c$, henc $Nint(Ncl((A))=U$, therefor A is NP-open set, so it is Nb-open set.
Hence we obtain $NPO(U,X)=NbO(U,X)=\{\varphi\} \cup \{A \subseteq U: A \cap U_R(X) \neq \emptyset\}$.

Corollary 3.26 In a nano topological space $(U,\tau_R(X))$, we have $NPO(U,X)=NbO(U,X)=N\beta O(U,X)=P(U)$ in these cases:
Case 1. where $U_R(X)=U$.

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Case 2. Where $L_R(X) = \varphi$ and $U_R(X) \neq U$.
Case 3. where $U_R(X) = L_R(X) = X$.
Proof. Direct from Theorems (3.2) and (3.25).

Theorem 3.27 [6] In a nano space $(U, \tau_R(X))$ if A and B are two Nb-open sets (NP-open sets) of U, then $A \cup B$ is also Nb-open set (NP-open set).

Example 3.28 The intersection of two Nb-open sets (NP-open sets) is not necessarily Nb-open set (NP-open set), for example in a nano topological space given in (3.6):
The sets$\{a, b, d, e\}$ and $\{b, c, d, e\}$ are Nb-open sets, but $\{b, d, e\} = \{a, b, d, e\} \cap \{b, c, d, e\}$ is not Nb-open.
The sets $\{a, b, d\}$ and $\{a, c, d\}$ are NP -open sets, but $\{a, d\} = \{a, b, d\} \cap \{a, c, d\}$ is not NP-open.

4. Conclusion

This article is devoted to review the class of generalized nano open sets, as nano α-open set, nano semi-open set, nano pre-open set, nano b-open set, nano β-open set and nano somewhere dense set, where we mentioned that nano β-open sets and nano somewhere dense sets are equivalent in nano topology, in addition we illustrated the relations between these class of sets, and then we studied the behavior of these generalization in some special cases regarding the lower and upper approximations, and proved that nano open set, nano α-open set and nano semi-open set are equivalent in three special cases, also nano pre-open set, nano b-open set and nano β-open set are equivalent in these cases.

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